Curvature Based Cooperative Exploration of Three Dimensional Scalar Fields

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***Abstract*— We develop strategies for controlled motion and filtering performed by a mobile sensor network to cooperatively explore an unknown 3D scalar field. In order to estimate the principal directions and principal curvatures of a desired level surface of the field which are used as feedback by the motion control law, we apply a non-analytic curvature estimation algorithm and prove the sufficient and necessary conditions under which this algorithm can provide reliable estimates. A differential geometric approach is followed in developing provably convergent steering control laws to control the center of the sensor platform formation to track one of the lines of curvature on a detected level surface of the scalar field.**

I. INTRODUCTION

Sensor platforms are deployed when explorative missions such as measuring the temperature or salinity in the ocean are encountered. To achieve mobility and adaptiveness, a cooperative group of mobile sensor platforms are expected to be superior to a single vehicle or a number of sensors that are fixed in the field [1], [2]. A combination of cooperative sensing and cooperative control is required for mission design. Zhang and Leonard [3] introduce strategies for multiple sensor platforms to track a level curve in a noisy planar scalar field. Other exploration activities such as climbing gradients of a scalar field [4], cooperative path following [1], [5] and monitoring environmental boundaries [6]–[8] are also introduced in the literature. Most known works focus on two-dimensional scalar fields. This paper develops strategies for mobile sensor networks to cooperatively explore a three dimensional scalar field.

We consider the mission of controlling the mobile sensor network to detect and track one of the lines of curvature on a level surface, where the principal directions and principal curvatures of the surface need to be estimated. The non-analytic curvature estimation algorithm introduced in [9] by Taubin, which was developed for computer vision applications, is applied in this context. An important concern in the cooperative exploration problem is the number of sensor platforms utilized to survey an area. Therefore, the minimum sensor quantity to implement Taubin's algorithm should be taken into consideration. Compared with analytic curvature estimation approaches that attempt to fit a smooth surface from a large amount of data, Taubin's algorithm can extract principal directions and principal curvatures directly from a relatively small amount of measurements taken near a surface. However, the algorithms described in [9] and other related works such as [10], [11] do not contain conditions for the minimum number of sensors required. We establish the sufficient and necessary conditions under which Taubin's algorithm can provide reliable results for estimating principal directions and principal curvatures in 3D cooperative exploration problems. In addition, we investigate constraints on the formation shape design.

A differential geometric approach is followed in developing steering control laws for 3D motions [12]–[14]. Following this approach, we use natural frames to describe the trajectory of the center of the sensor platform formation and one of the lines of curvature on a desired level surface. Based on Lyapunov stability criteria, we then design control laws to control the motion of the center to follow a desired line of curvature on the level surface.

This paper is organized as follows. In section II, we review the information dynamics of the cooperative exploration problem. Curvature estimation algorithms and the constraints on the sensor quantity are discussed in section III. In section IV, formation shape control using Jacobi vectors is derived. In section V, we develop control laws for the formation center motion control. Simulation results are shown in section VI. Concluding remarks is presented in section VII.

II. INFORMATION DYNAMICS

In this section, we review the information dynamic model for the cooperative exploration problem described in [3]. This model employs multiple sensor platforms that maintain a certain formation while exploring an unknown field. We use this model for the 3D cooperative exploration problem.

Assume that $z(\mathbf{r})$ is an unknown smooth field where $\mathbf{r} \in \mathbb{R}^3$. In real situations, the field is usually perturbed by noise and the measurements taken by sensor platforms are not perfect, which bring difficulties for a single sensor platform to explore. Therefore, multiple sensor platforms are employed to cooperatively estimate the field value.

Suppose we have N sensor platforms that take measurements discretely over time. Let the position of the ith sensor platform at time t_k be $\mathbf{r}_{i,k} \in \mathbb{R}^3$ and the value of the field at position $\mathbf{r}_{i,k}$ be $z(\mathbf{r}_{i,k})$, where $i = 1, 2, \cdots, N$. The measurement taken by the ith sensor platform is given by

$$p_{i,k} = z(\mathbf{r}_{i,k}) + w(\mathbf{r}_{i,k}) + n_{i,k}, \tag{1}$$

where $n_{i,k}$ are i.i.d Gaussian noise and $w(\mathbf{r}_{i,k})$ are spatially correlated Gaussian noise.

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Denote the position of the center of the formation at time t_k as $\mathbf{r}_{c,k}$. Using Taylor's expansion to approximate $z(\mathbf{r}_{i,k})$, we can get:

$$z(\mathbf{r}_{i,k}) \approx z(\mathbf{r}_{c,k}) + (\mathbf{r}_{i,k} - \mathbf{r}_{c,k})^T \nabla z(\mathbf{r}_{c,k}) + \frac{1}{2}(\mathbf{r}_{i,k} - \mathbf{r}_{c,k})^T \nabla^2 z(\mathbf{r}_{c,k})(\mathbf{r}_{i,k} - \mathbf{r}_{c,k}). \quad (2)$$

where $\nabla z(\mathbf{r}_{c,k})$ is the gradient and $\nabla^2 z(\mathbf{r}_{c,k})$ is the Hessian of the field.

Define $\mathbf{h}_{k-1} = (0, E[\mathbf{H}_{c,k}(\mathbf{r}_{c,k} - \mathbf{r}_{c,k-1})]^T)^T$ and $\mathbf{A}^s_{k-1} = \begin{pmatrix} 1 & (\mathbf{r}_{c,k} - \mathbf{r}_{c,k-1})^T \\ 0 & \mathbf{I}_{3\times 3} \end{pmatrix}$, where $\mathbf{H}_{c,k}$ is the estimate of the Hessian $\nabla^2 z(\mathbf{r}_{c,k})$ using the measurements taking by platforms at time t_k. Choose the state to be $\mathbf{s}_k = (z(\mathbf{r}_{c,k}), \nabla z(\mathbf{r}_{c,k})^T)^T$, then the state equation can be expressed as

$$\mathbf{s}_k = \mathbf{A}^s_{k-1}\mathbf{s}_{k-1} + \mathbf{h}_{k-1} + \varepsilon_{k-1}, \quad (3)$$

where ε_{k-1} is an $N \times 1$ noise vector.

Once the state equation (3) and measurement equation (1) are known, we can construct a cooperative Kalman filter to estimate the state of the system and reduce the measurement noise. The complete derivation of the cooperative Kalman filter can be found in [3].

III. Curvature Estimation

We consider a mission of controlling the center of the mobile sensor network to detect and track one of the lines of curvature on a desired level surface, which requires us to estimate principal curvatures and principal directions of this surface at each time instance, using measurements taken by N sensor platforms. In this section, we first briefly review some standard differential geometric terminologies [15], then we introduce the Taubin's algorithm and discuss its application in the 3D cooperative exploration problem. Our contribution is to determine the constraints on the sensor quantity and formation design.

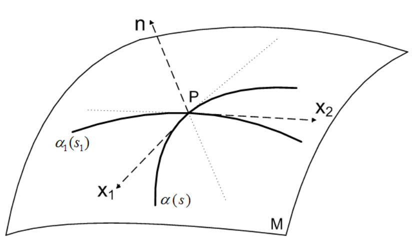

Fig. 1. Two curves on a level surface M. $\mathbf{x}_1$ and $\mathbf{x}_2$ are tangent vectors of $\alpha(s)$ and $\alpha_1(s_1)$. $\mathbf{n}$ is the normal vector of the surface at P.

A. Curvature Estimation Algorithm

As shown in Fig. 1, suppose $\alpha(s)$ and $\alpha_1(s_1)$ are two curves with arc-length parameters s and s_1 that intersect at point P on a smooth surface M. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be the tangent vectors of $\alpha(s)$ and $\alpha_1(s_1)$, and $\kappa_n(\mathbf{x}_1)$ and $\kappa_n(\mathbf{x}_2)$ be their corresponding normal curvatures. $\kappa_n(\mathbf{x}_1)$ and $\kappa_n(\mathbf{x}_2)$ are also known as the directional curvatures of the surface M at point P in directions of $\mathbf{x}_1$ and $\mathbf{x}_2$. Among all possible tangent directions at point P, if $\kappa_n(\mathbf{x}_1)$ takes the maximum value along $\mathbf{x}_1$ while $\kappa_n(\mathbf{x}_2)$ takes the minimum value along $\mathbf{x}_2$, then $\kappa_n(\mathbf{x}_1)$ and $\kappa_n(\mathbf{x}_2)$ are the two principal curvatures and $\mathbf{x}_1$ and $\mathbf{x}_2$ are the two corresponding principal directions of the surface M at P. In this case, $\mathbf{x}_1$ and $\mathbf{x}_2$ are perpendicular to each other and $\alpha(s)$, $\alpha_1(s_1)$ are called lines of curvature of the surface M. Note that the principal directions may not be unique for some smooth surfaces such as the sphere. We introduce the curvature estimation algorithm described

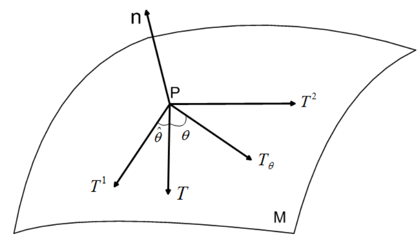

Fig. 2. $\mathbf{T}^1$ and $\mathbf{T}^2$ are the two principal directions of the surface at P. $\mathbf{T}$ and $\mathbf{T}_\theta$ are two arbitrarily chosen tangent vectors that form certain angles with $\mathbf{T}^1$. $\mathbf{n}$ is the normal vector to the surface at P.

by Taubin in [9] that is developed for computer vision applications. As shown in Fig. 2, let $\mathbf{T}^1$ and $\mathbf{T}^2$ denote the two principal directions of the surface M at point P with corresponding principal curvatures κ^1 and κ^2 where $\kappa^1 > \kappa^2$. $\mathbf{T}^1, \mathbf{T}^2, \kappa^1$ and κ^2 need to be estimated.

Choose an arbitrary unit tangent vector $\mathbf{T}$ to the surface at P that forms an angle $\hat{\theta}$ with $\mathbf{T}^1$ where $\hat{\theta}$ is unknown. For $-\pi < \theta < \pi$, define another unit tangent vector $\mathbf{T}_\theta$ to the surface at point P that forms an angle θ with $\mathbf{T}$. Let $\kappa_p(\mathbf{T}_\theta)$ be the directional curvature associated with the direction $\mathbf{T}_\theta$. Then a symmetric matrix $\mathbf{M}_p$ can be defined by an integral formula as

$$\mathbf{M}_p = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \kappa_p(\mathbf{T}_\theta)\mathbf{T}_\theta\mathbf{T}_\theta^T d\theta. \quad (4)$$

It can be shown that the principal directions $\mathbf{T}^1$, $\mathbf{T}^2$ and the normal vector $\mathbf{n}$ are eigenvectors of the matrix $\mathbf{M}_p$, which means that they can be computed by diagnolizing $\mathbf{M}_p$ as

$$\mathbf{M}_p = \begin{pmatrix} \mathbf{T}^1 & \mathbf{T}^2 & \mathbf{n} \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}^1 & \mathbf{T}^2 & \mathbf{n} \end{pmatrix}^T. \quad (5)$$

where λ_1 and λ_2 are the two non-zero eigenvalues of $\mathbf{M}_p$. It is further shown in [9] that the principal curvatures can be calculated as $\kappa^1 = 3\lambda_1 - \lambda_2$ and $\kappa^2 = 3\lambda_2 - \lambda_1$.

In order to apply Taubin's algorithm to the 3D cooperative exploration problem, we arrange sensor platforms so that at each time instance, they can partition the level surface that is passing through the center of the formation into a polyhedron. An example with eight sensor platforms is illustrated in Fig. 3. M is a level surface passing through the center of the formation $\mathbf{r}_c$. $\mathbf{r}_7$ and $\mathbf{r}_8$ are positions of the two sensor platforms located in the normal direction of the surface M at $\mathbf{r}_c$. All other sensor platforms $\mathbf{r}_1, ..., \mathbf{r}_6$ are lying in the tangent plane of M at $\mathbf{r}_c$. Along the positive or negative directions of $\mathbf{n}$, we can find $\mathbf{r}'_1, ..., \mathbf{r}'_6$, which divide the surface into 6 triangular faces. The unit vectors $\mathbf{T}_i, i = 1, ..., 6$ represent the projections of the vectors $\mathbf{r}_i - \mathbf{r}_c$ to the tangent plane of the the surface M at $\mathbf{r}_c$. The distance between each pair of sensor platforms is controlled by the formation shape control law that will be introduced in section IV. If we have N sensor platforms arranged in a similar way,

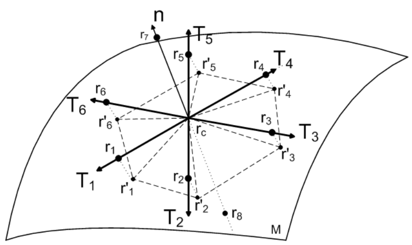

Fig. 3. Estimate principal directions and principal curvatures using eight sensor platforms. $\mathbf{r}_c$ is the center of the formation. $\mathbf{r}'_1, ..., \mathbf{r}'_6$ are points on the level surface obtained by searching along the negative or positive direction of the normal vector $\mathbf{n}$ from $\mathbf{r}_1, ..., \mathbf{r}_6$.

the approximation of the matrix $\mathbf{M}_p$ in (4) is

$$\mathbf{M}_v = \sum_{i=1}^{N-2} \omega_i \kappa_i \mathbf{T}_i \mathbf{T}_i^T, \tag{6}$$

where ω_i are the weighting factors that depend on the triangular areas determined by $\mathbf{r}_c, \mathbf{r}_1, ..., \mathbf{r}_{N-2}$ and satisfy the constraint $\sum \omega_i = 1$. κ_i is the directional curvature associated with $\mathbf{T}_i$ and is approximated by letting $\kappa_i = \frac{2\mathbf{n}^T(\mathbf{r}'_i - \mathbf{r}_c)}{\|\mathbf{r}'_i - \mathbf{r}_c\|}$.

From these calcuations, principal directions and principal curvatures of M at $\mathbf{r}_c$ can be estimated by diagonalizing $\mathbf{M}_v$.

B. Constraints on Sensor Quantity and Formation Design

When implementing Taubin's algorithm, we approximate the integral formula $\mathbf{M}_p$ with a finite sum $\mathbf{M}_v$. The number of sensor platforms and the way they are arranged will affect the estimation accuracy. On the other hand, an important concern in cooperative exploration problems is the minimum number of sensor platforms that can be utilized to navigate an area while providing reliable estimates.

Given a smooth surface M in three dimensional space, for $\mathbf{r}_c \in M$, assume that there exist two unique principal directions $\mathbf{T}^1 \in T_rM$ and $\mathbf{T}^2 \in T_rM$ where T_rM is the tangent plane of M at $\mathbf{r}_c \in M$. Define a set $\Omega = \{\mathbf{T} | \mathbf{T} \in T_rM, \mathbf{T} \neq \mathbf{T}^1, \mathbf{T} \neq \mathbf{T}^2, \|\mathbf{T}\| = 1\}$.

Suppose we have N sensor platforms arranged in a similar formation as illustrated in Fig. 3. From $N-2$ sensor platforms lying on the tangent plane of the surface M at $\mathbf{r}_c$, we arbitrarily select one as $\mathbf{r}_1$ and label others $\mathbf{r}_2, ...\mathbf{r}_{N-2}$ sequentially. The projection of the vector $\mathbf{r}_i - \mathbf{r}_c$ to the tangent plane of the surface M at $\mathbf{r}_c$ is $\mathbf{T}_i$. Denote the angle between the vector $\mathbf{T}_i$ with $\mathbf{T}_1$ as θ_i. Under this setting, $\theta_1 = 0$. We assume that the tangent vector $\mathbf{T}_1$ is selected so that $\mathbf{T}_1 \in \Omega$. If this is not true i.e. $\mathbf{T}_1$ is aligned with one of the principal directions, we assume that we can relabel the platforms so that another tangent vector that is not aligned with the principal directions is selected as $\mathbf{T}_1$. To satisfy this request, we see that $N \geq 4$.

With this configuration, we propose the following proposition.

Proposition 1: Consider a formation described above as illustrated in Fig. 3 with the assumptions that $\mathbf{T}_1 \in \Omega$ and that M has two principal directions. The Taubin's algorithm can provide estimates of principal curvatures and principal directions if and only if

$$\sum_{i=1}^{N-2} \omega_i \kappa_i \sin 2\theta_i \neq 0 \tag{7}$$

where θ_i is the angle between $\mathbf{T}_i$ and $\mathbf{T}_1$ and $\theta_1 = 0$.

Proof: Choose $\mathbf{T}_1$ and the corresponding orthonormal vector $\mathbf{T}_1^\perp$ as the basis of the tangent plane, then $\mathbf{T}_i$ can be written as:

$$\mathbf{T}_i = \mathbf{T}_1 \cos\theta_i + \mathbf{T}_1^\perp \sin\theta_i, \; i = 1, 2, ..., N-2. \tag{8}$$

Therefore,

$$\begin{aligned} \mathbf{T}_i \mathbf{T}_i^T = & \mathbf{T}_1 \mathbf{T}_1^T \cos^2\theta_i + \mathbf{T}_1 (\mathbf{T}_1^\perp)^T \cos\theta_i \sin\theta_i \\ & + \mathbf{T}_1^\perp \mathbf{T}_1^T \cos\theta_i \sin\theta_i + \mathbf{T}_1^\perp (\mathbf{T}_1^\perp)^T \sin^2\theta_i. \end{aligned} \tag{9}$$

Denote $\omega_i \kappa_i = \kappa'_i$ and plug $\mathbf{T}_i \mathbf{T}_i^T$ into equation (6), we can obtain

$$\begin{aligned} \mathbf{M}_v = & \sum_{i=1}^{N-2} \kappa'_i (\mathbf{T}_1 \mathbf{T}_1^T \cos^2\theta_i + \mathbf{T}_1 (\mathbf{T}_1^\perp)^T \cos\theta_i \sin\theta_i \\ & + \mathbf{T}_1^\perp \mathbf{T}_1^T \cos\theta_i \sin\theta_i + \mathbf{T}_1^\perp (\mathbf{T}_1^\perp)^T \sin^2\theta_i). \end{aligned} \tag{10}$$

Suppose $\hat{\mathbf{T}}^1$ is one of the estimated principal directions that can be expressed as $\hat{\mathbf{T}}^1 = \mathbf{T}_1 \cos\hat{\theta} + \mathbf{T}_1^\perp \sin\hat{\theta}$ where $\hat{\theta} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ is the angle between $\hat{\mathbf{T}}^1$ and $\mathbf{T}_1$. Then according to Taubin's algorithm, we can write down the following relationship:

$$\mathbf{M}_v \hat{\mathbf{T}}^1 = \hat{\lambda}_1 \hat{\mathbf{T}}^1 = \mathbf{T}_1 \hat{\lambda}_1 \cos\hat{\theta} + \mathbf{T}_1^\perp \hat{\lambda}_1 \sin\hat{\theta} \tag{11}$$

where $\hat{\lambda}_1$ is the eigenvalue corresponding to $\hat{\mathbf{T}}^1$. On the other hand,

$$\mathbf{M}_v \hat{\mathbf{T}}^1 = \mathbf{M}_v (\mathbf{T}_1 \cos\hat{\theta} + \mathbf{T}_1^\perp \sin\hat{\theta}). \tag{12}$$

Substitute $\mathbf{M}_v$ in equation (10) into the above equation and use the relationship $\mathbf{T}_1^T \mathbf{T}_1 = (\mathbf{T}_1^\perp)^T \mathbf{T}_1^\perp = 1$ and $(\mathbf{T}_1^\perp)^T \mathbf{T}_1 = \mathbf{T}_1^T \mathbf{T}_1^\perp = 0$, $\mathbf{M}_v \hat{\mathbf{T}}^1$ can be calculated as

$$\begin{aligned} \mathbf{M}_v \hat{\mathbf{T}}^1 = & \mathbf{T}_1 [\sum_{i=1}^{N-2} \kappa'_i (\cos^2\theta_i \cos\hat{\theta} + \frac{1}{2} \sin 2\theta_i \sin\hat{\theta})] \\ & + \mathbf{T}_1^\perp [\sum_{i=1}^{N-2} \kappa'_i (\sin^2\theta_i \sin\hat{\theta} + \frac{1}{2} \sin 2\theta_i \cos\hat{\theta})]. \end{aligned} \tag{13}$$

Comparing equation (11) with equation (13), we can obtain

$$\begin{aligned} \hat{\lambda}_1 &= \sum_{i=1}^{N-2} \kappa'_i \cos^2\theta_i + \frac{1}{2} \sum_{i=1}^{N-2} \kappa'_i \sin 2\theta_i \tan\hat{\theta} \\ \hat{\lambda}_1 &= \sum_{i=1}^{N-2} \kappa'_i \sin^2\theta_i + \frac{1}{2} \sum_{i=1}^{N-2} \kappa'_i \sin 2\theta_i \cot\hat{\theta}. \end{aligned} \tag{14}$$

If $\sum_{i=1}^{N-2} \kappa_i' \sin 2\theta_i \neq 0$, the above two equations give well defined solutions for $\hat{\theta}$ that satisfy:

$$\tan^2 \hat{\theta} + \frac{2\sum_{i=1}^{N-2} \kappa_i' \cos 2\theta_i}{\sum_{i=1}^{N-2} \kappa_i' \sin 2\theta_i} \tan \hat{\theta} - 1 = 0. \tag{15}$$

For each solution $\hat{\theta}$, the estimated eigenvector $\hat{\mathbf{T}}^1$ has the form of $\mathbf{T}_1 \cos \hat{\theta} + \mathbf{T}_1^{\perp} \sin \hat{\theta}$. This finishes the proof for the sufficient condition.

From the relationship $\mathbf{T}_1^T \mathbf{T}_1 = 1$ and $(\mathbf{T}_1^{\perp})^T \mathbf{T}_1 = 0$, we also have

$$\mathbf{M}_v \mathbf{T}_1 = \mathbf{T}_1 \sum_{i=1}^{N-2} \kappa_i' \cos^2 \theta_i + \frac{1}{2} \mathbf{T}_1^{\perp} \sum_{i=1}^{N-2} \kappa_i' \sin 2\theta_i. \tag{16}$$

We now use proof by contradiction to show the necessary condition. Suppose the term $\sum_{i=1}^{N-2} \kappa_i' \sin 2\theta_i$ sums to zero, then

$$\mathbf{M}_v \mathbf{T}_1 = \mathbf{T}_1 \sum_{i=1}^{N-2} \kappa_i' \cos^2 \theta = \lambda_1 \mathbf{T}_1, \tag{17}$$

where λ_1 is a scalar. From the above equation, we can see that $\mathbf{T}_1$ is one of the eigenvectors of $\mathbf{M}_v$ and λ_1 is the corresponding eigenvalue. According to Taubin's algorithm, this results in $\mathbf{T}_1$ being one of the principal directions. However, $\mathbf{T}_1$ is not aligned with any principal directions since $\mathbf{T}_1 \in \Omega$. This contradiction means that Taubin's algorithm can produce estimates of principal directions only if $\sum_{i=1}^{N-2} \kappa_i' \sin 2\theta_i \neq 0$.

■

We now give an example where (7) is violated. Consider a symmetric formation with N sensor platforms, the angle between $\mathbf{T_i}$ and $\mathbf{T_1}$ is $\theta_i = \frac{2\pi}{N-2}(i-1)$. Plug θ_i into $\sum_{i=1}^{N-2} \kappa_i' \sin 2\theta_i$, we can obtain

$$\sum_{i=1}^{N-2} \kappa_i' \sin 2\theta_i = \sum_{i=1}^{N-2} \kappa_i' \sin(\frac{4\pi}{N-2}(i-1)). \tag{18}$$

When $N = 4$, equation (18) can be written as

$$\kappa_1' \sin 0 + \kappa_3' \sin 2\pi = 0. \tag{19}$$

Similarly, when $N = 6$, we have

$$\kappa_1' \sin 0 + \kappa_2' \sin \pi + \kappa_3' \sin 2\pi + \kappa_4' \sin 3\pi = 0. \tag{20}$$

The summations will be zero regardless of the labeling of the sensor platforms, which means that we can not use Taubin's algorithm to estimate principal curvatures and principal directions with four or six sensor platforms arranged in a symmetric way.

IV. FORMATION SHAPE CONTROL

In order to keep all sensor platforms in a desired formation, we apply the similar control law introduced in [3] that describes the shape of the formation using Jacobi vectors. Jacobi transform is a powerful method to decouple the dynamics of the formation shape and formation center which allows us to design decoupled control laws.

With N sensor platforms assumed to have unit mass, the $N-1$ Jacobi vectors used to describe the formation satisfy the following criterion: $[\mathbf{r}_c, \mathbf{q}_1, \cdots, \mathbf{q}_{N-1}] = [\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N]\Psi$ where Ψ is the Jacobi transform matrix [16].

The Newton's equation for each sensor is $\ddot{\mathbf{r}}_i = \mathbf{f}_i$ where $\mathbf{f}_i$ is the force for the ith sensor. When Jacobi vectors are used, the equations become $\ddot{\mathbf{q}}_j = \mathbf{u}_j$ and $N\ddot{\mathbf{r}}_c = \mathbf{f}_c$ where $\mathbf{u}_j$ and $\mathbf{f}_c$ are forces for jth Jacobi vector and the formation center.

Let $\mathbf{q}_j^0$ be the desired vectors that define a certain formation. In [3], $\mathbf{q}_j^0$ are chosen to be constant in order to remain an unchanging formation. The control force $\mathbf{u}_j$ for $\mathbf{q}_j$ should be designed that $\mathbf{q}_j$ converges to $\mathbf{q}_j^0$. We use the similar form of control law as in the 2D case here: $\mathbf{u}_j = -K_1(\mathbf{q}_j - \mathbf{q}_j^0) - K_2 \dot{\mathbf{q}}_j$, where K_1 and K_2 are positive gains. It can be proved that under this control law, the sensor platforms converge to the desired formation with an exponential rate of convergence.

V. FORMATION CENTER CONTROL

With the principal directions and principal curvatures estimated using methods discussed in section III, we develop control laws governing the center of the platform formation to track a line of curvature on a level surface using a differential geometric approach.

A. Dynamic equations

At any time instant, there is a level surface passing through the center of the formation $\mathbf{r}_c$. We assume that, at any point of this level surface, there exist two unique principal directions and principal curvatures. The formation will be controlled to track the line of curvature associated with the larger principal curvature.

Let $\mathbf{r}$ denote the point on the curve that the center of the formation should track. $\mathbf{x}_1$ is the unit tangent vector to the curve at point $\mathbf{r}$, $\mathbf{n}$ is the unit normal vector and $\mathbf{x}_2$ is the binormal vector. The velocity of the point is in the direction of $\mathbf{x}_1$. With the speed defined by $\frac{ds}{dt} = \alpha$, the equations that describes the time-evolution of the point on the curve are

$$\begin{aligned} \dot{\mathbf{r}} &= \alpha \mathbf{x}_1 \\ \dot{\mathbf{x}}_1 &= \alpha \kappa_n \mathbf{n} + \alpha \kappa_g \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= -\alpha \kappa_g \mathbf{x}_1 \\ \dot{\mathbf{n}} &= -\alpha \kappa_n \mathbf{x}_1. \end{aligned} \tag{21}$$

where κ_n and κ_g are the normal curvature and the geodesic curvature of the curve respectively. Since the desired curve is a line of curvature, $\mathbf{x}_1$ and $\mathbf{x}_2$ should be aligned with the principal directions and κ_n should be the larger principal curvature estimated in section III.

Let $\mathbf{r}_c$ denote the center of the formation moving at unit speed. $\mathbf{X}_1$ is the unit tangent vector to the trajectory of the moving center, $\mathbf{N}$ is the unit normal vector and $\mathbf{X}_2$ is given by $\mathbf{N} \times \mathbf{X}_1$. We have

$$\begin{aligned} \dot{\mathbf{r}}_c &= \mathbf{X}_1 \\ \dot{\mathbf{X}}_1 &= u\mathbf{N} + v\mathbf{X}_2 \\ \dot{\mathbf{X}}_2 &= -v\mathbf{X}_2 \\ \dot{\mathbf{N}} &= -u\mathbf{X}_1. \end{aligned} \tag{22}$$

where u and v are steering controls we will design for the moving formation. The goal is to control the center of the formation $\mathbf{r}_c$ to track $\mathbf{r}$, and to control $\mathbf{X}_1$ to be aligned with $\mathbf{x}_1$.

B. Steering Controller Design

Let z_c be the value of the formation center estimated by the cooperative Kalman filter using the measurements from all sensor platforms. With the relationship $\mathbf{n} = \frac{\nabla z(\mathbf{r(s)})}{\|\nabla z(\mathbf{r}(s))\|}$, when the formation is moving on the level surface, the value z_c is changing with respect to time:

$$\dot{z}_c = \nabla z \cdot \frac{d\mathbf{r}_c}{dt} = \nabla z \cdot \mathbf{X}_1 = \|\nabla z\| \mathbf{n} \cdot \mathbf{X}_1. \tag{23}$$

Assume that the field has extrema $z_{min} < z_{max}$. Consider a Lyapunov candidate function which is analogous to the one chosen in [17] and [13]:

$$V = -\ln \mathbf{X}_1 \cdot \mathbf{x}_1 + h(z_c), \tag{24}$$

where $h(z_c)$ satisfies the following assumptions:

1. $h(z_c)$ is continuously differentiable on (z_{min}, z_{max}). $f(z_c) = \frac{dh}{dz_c}$ is a Lipschitz continuous function.

2. $f(C) = 0$ and $f(z) \neq 0$ if $z \neq C$ where C is the desired level surface value.

3. $\lim_{z \to z_{min}} h(z) = \infty, \lim_{z \to z_{max}} h(z) = \infty$ and $\exists \tilde{z}$ such that $h(\tilde{z}) = 0$.

The term $\ln \mathbf{X}_1 \cdot \mathbf{x}_1$ in the Lyapunov function helps align the moving direction of the formation center with the tangent vector of the line of curvature passing through the center. In fact, if we initially set $\mathbf{X}_1 \cdot \mathbf{x}_1 > 0$, then $\mathbf{X}_1 \cdot \mathbf{x}$ will stay larger than 0. The other term $h(z_c)$ serves to control the formation to stay on the desired level surface.

The control law can be designed as

$$\begin{aligned} u &= (f(z_c) \parallel \nabla z \parallel \mathbf{n} + \frac{\alpha \kappa_n}{\mathbf{X}_1 \cdot \mathbf{x}_1} \mathbf{n} + \frac{\alpha \kappa_g}{\mathbf{X}_1 \cdot \mathbf{x}_1} \mathbf{x}_2 + \mu \mathbf{x}_1) \cdot \mathbf{N}, \\ v &= (f(z_c) \parallel \nabla z \parallel \mathbf{n} + \frac{\alpha \kappa_n}{\mathbf{X}_1 \cdot \mathbf{x}_1} \mathbf{n} + \frac{\alpha \kappa_g}{\mathbf{X}_1 \cdot \mathbf{x}_1} \mathbf{x}_2 + \mu \mathbf{x}_1) \cdot \mathbf{X}_2. \end{aligned} \tag{25}$$

Under these control laws, we can derive the closed loop dynamics that govern the relative motion between the formation center $\mathbf{r}_c$ and the point $\mathbf{r}$ on the desired curve. For this purpose we define three shape variables as $\varphi = \mathbf{x}_1 \cdot \mathbf{X}_1, \beta = \mathbf{n} \cdot \mathbf{X}_1$ and $\gamma = \mathbf{x}_2 \cdot \mathbf{X}_1$. It can be shown that the closed loop dynamics are

$$\begin{aligned} \dot{\varphi} &= \mu(\beta^2 + \gamma^2) - f(z_c) \|\nabla z\| \beta \varphi \\ \dot{\beta} &= -\alpha \kappa_n \varphi + (f(z_c) \|\nabla z\| + \frac{\alpha \kappa_n}{\varphi})(\varphi^2 + \gamma^2) - \frac{\alpha \kappa_g}{\varphi} \beta \gamma - \mu \beta \varphi \\ \dot{\gamma} &= -\alpha \kappa_g \varphi - (f(z_c) \|\nabla z\| + \frac{\alpha \kappa_n}{\varphi}) \beta \gamma + \frac{\alpha \kappa_g}{\varphi} (\varphi^2 + \beta^2) - \mu \varphi \gamma \\ \dot{z}_c &= \|\nabla z\| \beta. \end{aligned} \tag{26}$$

If κ_n, κ_g and $\|\nabla z\|$ are bounded for the curve that the formation center is tracking, the above system is a well defined time-varying nonlinear system. Then the Lyapunov candidate function V becomes $V = -\ln \varphi + h(z_c)$ that is defined on the state space of $(\varphi, \beta, \gamma, z_c)$ where $\varphi^2 + \beta^2 + \gamma^2 = 1$.

Proposition 2: Consider a smooth scalar field with bounded Hessian and bounded gradient that satisfies $\|\nabla z(\mathbf{r})\| \neq 0$ except for a finite number of points $\mathbf{r}_{sup}$ where $z(\mathbf{r}_{sup}) = z_{min}$ or $z(\mathbf{r}_{sup}) = z_{max}$. Under the control law in equation (25), we will have $\mathbf{X}_1$ aligned with $\mathbf{x}_1$ and $z_c \to C$ asymptotically if the initial value $\mathbf{X}_1 \cdot \mathbf{x}_1 > 0$ and $\mathbf{r}(t_0) \neq \mathbf{r}_{sup}$.

Proof: The derivative of the Lyapunov candidate function V is

$$\begin{aligned} \dot{V} &= -\frac{1}{\varphi} \dot{\varphi} + f(z_c) \dot{z}_c \\ &= -\frac{1}{\varphi} (\mu(\beta^2 + \gamma^2) - f(z_c) \|\nabla z\| \beta \varphi) + f(z_c) \|\nabla z\| \beta \\ &= -\frac{\mu}{\varphi} (1 - \varphi^2) \leq 0 \end{aligned} \tag{27}$$

Since $V \to \infty$ as (1) $\varphi \to 0$ and (2) $z_c \to z_{max}$ or $z_c \to z_{min}$, if the trajectory initially satisfies $\varphi > 0$ and $z_c \in [z_{min}, z_{max}]$, then the trajectory will stay in a compact sub-level set of the Lyapunov function V. Let E be the set within the sub-level set where $\dot{V} = 0$ i.e. $E = \{(\varphi, \beta, \gamma, z_c) | \varphi = 1, \beta = 0, \gamma = 0\}$. Because the closed loop system (26) is time-varying, we can not apply the classical LaSalle's Invariance Principle. Instead, a more advanced invariance theorem can be applied [18] to claim that the trajectory will converge to the set E when $t \to \infty$.

At points in E, the closed loop system becomes

$$\begin{aligned} \dot{z}_c &= 0 \\ \dot{\varphi} &= 0 \\ \dot{\beta} &= f(z_c) \|\nabla z\| \\ \dot{\gamma} &= 0. \end{aligned} \tag{28}$$

According to the Barbalat Lemma [18], if $f(z_c) \|\nabla z\|$ is uniformly continuous and $\beta = 0$, then $\dot{\beta} = 0$ must hold. In fact, since z_c is constant, $\|\nabla z\|$ is assumed to be bounded, and the vector field is assumed to have smooth level curves, it can be shown that $\|\nabla z\|$ is uniformly continuous along level curves. Therefore, we conclude that $f(z_c) \|\nabla z\| = 0$, which implies $f(z_c) = 0$ on E. This means the trajectory of the formation center will be aligned with the principal directions and the field value at the formation center will converge to the desired constant value C. ■

VI. SIMULATION RESULTS

We first demonstrate the cooperative exploration strategy we developed using eight sensor platforms. The unknown field is assumed to be composed of several cylindric level surfaces. Suppose the field value is well measured, that is, when the position of each sensor platform is determined by control laws, we can have the accurate measurement of the field value at that position. For a cylinder, lines of curvature with maximum principal curvatures correspond to a set of circumferences of lateral surfaces. As shown in Fig. 4, green circles are lines of curvatures of the desired level surface $C = 1$. At each step, the six sensor platforms in the tangent plane take measurements at the same time while moving, then give estimates of principal directions and principal curvatures using Taubin's algorithm. The initial position of the formation center is -0.2 off the level surface. Under the motion control law, the center of formation converges to the desired level surface and track a line of curvature. In this case, the curvature estimation algorithm gives correct estimate of principal directions and principal curvatures.

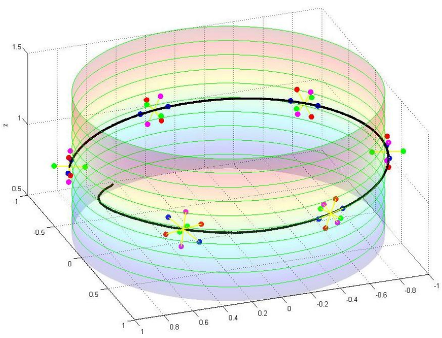

Fig. 4. Simulation result using eight sensor platforms. The desired level surface value is 1. Green curves are lines of curvatures of the level surface. Black line is the trajectory of the formation center. Eight sensor platforms are controlled to track one of lines of curvatures and converge to the desired formation.

Another experiment is conducted using six sensor platforms with four of them lying in the tangent plane, in a symmetric formation. Other settings are the same as the first experiment. As shown in Fig. 5, the formation failed to track one of lines of curvature of the surface even though it is still controlled in the level surface. This simulation testifies the fact that when six sensor platforms are deployed, they should not be arranged in a symmetric formation.

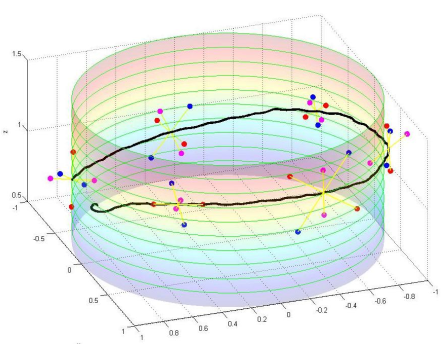

Fig. 5. Simulation result using six sensor platforms. Six sensor platforms failed to track a line of curvature.

VII. CONCLUSIONS AND FUTURE WORKS

We have developed strategies for a mobile sensor network to cooperatively explore a scalar field in 3D. Taubin's algorithm for estimating principal directions and principal curvatures has been implemented using the measurements taken by sensor platforms near the surface. We have proved the sufficient and necessary conditions on certain arrangements of sensor platforms for the algorithm to give reliable estimates. In order to allow the center of the formation to track one of the lines of curvature on a desired level surface, we have developed three dimensional steering control laws with provable convergence using a differential geometric approach.

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